# Решение задачи Коши для дифференциального уравнения в частных производных

ЗАДАНИЕ.

Решить задачу Коши для уравнения в частных производных

$$u_{tt} - 2\Delta u = (x^2 + y^2 + z^2)t; \quad u\big|_{t=0} = xyz, \quad u_t\big|_{t=0} = x - y.$$

РЕШЕНИЕ.

Здесь $\overline{x} = (x, y, z), \quad u_0 = xyz, \quad u_1 = x - y.$

Так как $a^2 = 2$ и $f(\overline{x}, t) = \left(x^2 + y^2 + z^2\right)t$, то по (47)

$$u_{n+2}(\overline{x}) = 2\Delta u_n(\overline{x}) + \frac{\partial^n f(\overline{x}, t)}{\partial t^n}\Big|_{t=0} \quad , n = 0,1,2,...$$

Отсюда

$$u_2 = 2\Delta u_0 + f(\overline{x}, t)\big|_{t=0} = 2\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial z^2}\right) + 0 = 2\left(0+0+0\right) + 0 = 0,$$

$$u_3 = 2\Delta u_1 + \frac{\partial f(\overline{x}, t)}{\partial t}\Big|_{t=0} = 2\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) + \left(x^2 + y^2 + z^2\right) =$$

$$= 2\left(0+0+0\right) + x^2 + y^2 + z^2 = x^2 + y^2 + z^2,$$

$$u_4 = 2\Delta u_2 + \frac{\partial^2 f(\overline{x}, t)}{\partial t^2}\Big|_{t=0} = 2\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}\right) + 0 = 0,$$

$$u_5 = 2\Delta u_3 + \frac{\partial^3 f(\overline{x}, t)}{\partial t^3}\Big|_{t=0} = 2\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial^2 u_3}{\partial z^2}\right) + 0 = 4,$$

$$u_6 = 2\Delta u_4 + \frac{\partial^4 f(\overline{x}, t)}{\partial t^4}\Big|_{t=0} = 0,$$

$$u_7 = 2\Delta u_5 + \frac{\partial^5 f(\overline{x}, t)}{\partial t^5}\Big|_{t=0} = 0, u_8, u_9, ... = 0.$$

То есть, все остальные $u_{2k} = 0 \ (k \geq 3)$ è $u_{2k+1} = 0 \ (k \geq 3)$.

Подставляем найденные $u_n$ в решение (46)

$$u(\overline{x}, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} u_n(\overline{x}) = u_0 + t u_1 + \frac{t^2}{2!} u_2 + \frac{t^3}{3!} u_3 + \frac{t^4}{4!} u_4 + \frac{t^5}{5!} u_5 + ... \quad ,$$

$$u(x,y,z,t)=xyz+\left(x-y\right)t+\frac{t^3}{6}\left(x^2+y^2+z^2\right)+\frac{t^5}{120}\cdot 4=$$

$$=xyz+\left(x-y\right)t+\frac{t^3}{6}\left(x^2+y^2+z^2\right)+\frac{t^5}{30}.$$

Окончательно,

$$u(x,y,z,t)=xyz+\left(x-y\right)t+\frac{t^3}{6}\left(x^2+y^2+z^2\right)+\frac{t^5}{30}..$$