# Пример решения задачи: тройной интеграл двумя разными заменами

Задание.

*Решить тройной интеграл двумя способами (цилиндрическая и сферическая замена координат)*

$$\iiint\limits_G (x^2+y^2+z^2)^2 dxdydz,$$

$$\text{где } G=\{(x,y,z): x^2+y^2+z^2 \le a^2, x+z \ge 0\}$$

Решение.

<u>Цилиндрическая система координат.</u>

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}; \quad dxdydz = r\,drd\varphi dz$$

Выражение $x^2+y^2+z^2 \le a^2$ примет вид $r^2+z^2 \le a^2$, тогда

$$G: \begin{cases} -r\cos\varphi \le z \le \sqrt{a^2-r^2} \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$$

Интеграл примет вид:

$$\iiint\limits_G (x^2+y^2+z^2)^2 dxdydz = \int_0^{2\pi} d\varphi \int_0^a dr \int_{-r\cos\varphi}^{\sqrt{a^2-r^2}} (r^2+z^2)^2 \cdot r dz$$

$$\int_{-r\cos\varphi}^{\sqrt{a^2-r^2}} (r^2+z^2)^2 \cdot r dz = r \int_{-r\cos\varphi}^{\sqrt{a^2-r^2}} (r^4+2r^2z^2+z^4) dz$$

$$= r\left(r^4 z + \frac{2r^2}{3}z^3 + \frac{1}{5}z^5\right)\Bigg|_{-r\cos\varphi}^{\sqrt{a^2-r^2}} =$$

$$= r^5\sqrt{a^2-r^2} + \frac{2r^3}{3}\sqrt{(a^2-r^2)^3} + \frac{1}{5}r\sqrt{(a^2-r^2)^5} + r^6\cos\varphi + \frac{2}{3}r^6\cos^3\varphi + \frac{1}{5}r^6\cos^5\varphi$$

$$\int_0^a r^5\sqrt{a^2-r^2}dr = \left|\begin{matrix} t^2=a^2-r^2 & r=\sqrt{a^2-t^2} & r=a\to t=0 \\ r^2=a^2-t^2 & dr=-\dfrac{t}{\sqrt{a^2-t^2}}dt & r=0\to t=a \end{matrix}\right| =$$

$$= \int_a^0 \left(\sqrt{a^2-t^2}\right)^5 \cdot t \cdot \left(-\frac{t}{\sqrt{a^2-t^2}}\right)\cdot dt = \int_0^a (a^2-t^2)^2\cdot t^2 dt$$

$$= \int_0^a (a^4t^2-2a^2t^4+t^6)dt =$$

$$= \left(\frac{a^4}{3}t^3-\frac{2a^2}{5}t^5+\frac{1}{7}t^7\right)\Big|_0^a = \frac{1}{3}a^7-\frac{2}{5}a^7+\frac{1}{7}a^7 = \frac{8}{105}a^7$$

$$\int_0^a r^3\sqrt{(a^2-r^2)^3}dr = \left|\begin{matrix} t^2=a^2-r^2 & r=\sqrt{a^2-t^2} & r=a\to t=0 \\ r^2=a^2-t^2 & dr=-\dfrac{t}{\sqrt{a^2-t^2}}dt & r=0\to t=a \end{matrix}\right| =$$

$$= \int_a^0 \left(\sqrt{a^2-t^2}\right)^3 \cdot t^3 \cdot \left(-\frac{t}{\sqrt{a^2-t^2}}\right)\cdot dt = \int_0^a (a^2-t^2)\cdot t^4 dt$$

$$= \int_0^a (a^2t^4-t^6)dt =$$

$$= \left(\frac{a^2}{5}t^5-\frac{1}{7}t^7\right)\Big|_0^a = \frac{1}{5}a^7-\frac{1}{7}a^7 = \frac{2}{35}a^7$$

$$\int_0^a r\sqrt{(a^2-r^2)^5}dr = \left|\begin{matrix} t^2=a^2-r^2 & r=\sqrt{a^2-t^2} & r=a\to t=0 \\ r^2=a^2-t^2 & dr=-\dfrac{t}{\sqrt{a^2-t^2}}dt & r=0\to t=a \end{matrix}\right| =$$

$$= \int_a^0 \sqrt{a^2-t^2}\cdot t^5 \cdot \left(-\frac{t}{\sqrt{a^2-t^2}}\right)\cdot dt = \int_0^a t^6 dt = \frac{1}{7}t^7\Big|_0^a = \frac{1}{7}a^7$$

$$\int_0^a \left(r^6\cos\varphi + \frac{2}{3}r^6\cos^3\varphi + r^6\cos^5\varphi\right)dr$$

$$= \frac{1}{7}\left(\cos\varphi+\frac{2}{3}\cos^3\varphi+\frac{1}{5}\cos^5\varphi\right)r^7\Big|_0^a =$$

$$= \frac{1}{7}a^7\left(\cos\varphi+\frac{2}{3}\cos^3\varphi+\frac{1}{5}\cos^5\varphi\right)$$

$$\int_0^{2\pi}\left(\frac{8}{105}a^7+\frac{2}{3}\cdot\frac{2}{35}a^7+\frac{1}{5}\cdot\frac{1}{7}a^7+\frac{1}{7}a^7\left(\cos\varphi+\frac{2}{3}\cos^3\varphi+\frac{1}{5}\cos^5\varphi\right)\right)d\varphi=$$

$$=\frac{1}{7}a^7\int_0^{2\pi}\left(1+\cos\varphi+\frac{2}{3}\cos^3\varphi+\frac{1}{5}\cos^5\varphi\right)d\varphi=\frac{1}{7}a^7(\varphi+\sin\varphi)\Big|_0^{2\pi}+$$

$$+\frac{2}{21}a^7\int_0^{2\pi}(1-\sin^2\varphi)d(\sin\varphi)+\frac{1}{35}a^7\int_0^{2\pi}(1-\sin^2\varphi)^2d(\sin\varphi)=\frac{2}{7}\pi a^7+$$

$$+\frac{1}{7}a^7(\sin 2\pi-\sin 0)+\frac{2}{21}a^7\left(\sin\varphi-\frac{1}{3}\sin^3\varphi\right)\Big|_0^{2\pi}$$

$$+\frac{1}{35}a^7\left(\sin\varphi-\frac{2}{3}\sin^3\varphi-\frac{1}{5}\sin^5\varphi\right)\Big|_0^{2\pi}=$$

$$=\frac{2}{7}\pi a^7$$

<u>Сферическая система координат.</u>

$$\begin{cases} x=r\cos\varphi\sin\theta \\ y=r\sin\varphi\sin\theta\,; \quad dxdydz=r^2\sin\theta\ drd\varphi d\theta \\ z=r\cos\theta \end{cases}$$

Выражение $x^2+y^2+z^2\le a^2$ примет вид $r\le a$, выражение $x+z\ge 0$:

$$r\cos\varphi\sin\theta+r\cos\theta\ge 0$$

$$\cos\varphi\sin\theta+\cos\theta\ge 0$$

$$\cos\varphi\ge-\operatorname{ctg}\theta\,;\quad \theta\ge-\operatorname{arcctg}(\cos\varphi)$$

$$G:\begin{cases} 0\le r\le a \\ -\operatorname{arcctg}(\cos\varphi)\le\theta\le\pi \\ 0\le\varphi\le 2\pi \end{cases}$$

Интеграл примет вид:

$$\iiint_G(x^2+y^2+z^2)^2dxdydz=\int_0^{2\pi}d\varphi\int_{-\operatorname{arcctg}(\cos\varphi)}^{\pi}d\theta\int_0^a r^4\cdot r^2\sin\theta\,dr$$

$$\int_0^a r^4\cdot r^2\sin\theta\,dr=\frac{1}{7}r^7\sin\theta\Big|_0^a=\frac{1}{7}a^7\sin\theta$$

$$\int_{-\operatorname{arcctg}(\cos\varphi)}^{\pi} \frac{1}{7}a^7 \sin\theta\, d\theta = -\frac{1}{7}a^7\cos\theta \Big|_{-\operatorname{arcctg}(\cos\varphi)}^{\pi}$$

$$= -\frac{1}{7}a^7(\cos\pi - \cos(-\operatorname{arcctg}(\cos\varphi))) =$$

$$= -\frac{1}{7}a^7\left(-1 - \frac{\cos\varphi}{\sqrt{1+\cos^2\varphi}}\right) = \frac{1}{7}a^7\left(1 + \frac{\cos\varphi}{\sqrt{1+\cos^2\varphi}}\right)$$

$$\int_0^{2\pi} \frac{1}{7}a^7\left(1 + \frac{\cos\varphi}{\sqrt{1+\cos^2\varphi}}\right)d\varphi = \frac{1}{7}a^7\varphi\Big|_0^{2\pi} + \frac{1}{7}a^7\int_0^{2\pi}\frac{1}{\sqrt{2-\sin^2\varphi}}d(\sin\varphi)$$

$$= \frac{2}{7}a^7\pi +$$

$$+\frac{1}{7}a^7\arcsin\left(\frac{\sin\varphi}{\sqrt{2}}\right)\Big|_0^{2\pi} = \frac{2}{7}a^7\pi + \frac{1}{7}a^7\left(\arcsin\left(\frac{\sin 2\pi}{\sqrt{2}}\right) - \arcsin\left(\frac{0}{\sqrt{2}}\right)\right)$$

$$= \frac{2}{7}a^7\pi$$

ОТВЕТ. $\frac{2}{7}a^7\pi$